Pascal's Triangle, Part I: The Visible Architecture of the Triangle

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Abstract

Pascal's triangle is one of the most familiar objects in elementary mathematics, yet it contains a surprising wealth of structure. Before turning to its algebraic role in binomial expansion, it is worthwhile to study the triangle simply as a pattern of numbers built from a very simple rule. In this article, we examine its visible architecture: the boundary entries, the symmetry of each row, and the remarkable sequences that arise along its shallow diagonals. Although these patterns are easy to observe, they are not merely accidental. Each follows from the construction of the triangle itself.

Introduction

Pascal's triangle may be written as follows:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

The triangle is built according to a simple rule:

- the first and last entry in each row are 1;
- every interior entry is obtained by adding the two entries directly above it.

Throughout this article, the top row will be called row 0. We shall denote by $P_{n,k}$ the entry in row n and position k, counting from the left, where $0 \leq k \leq n$. Thus the construction rule may be written as

$$P_{n,0} = P_{n,n} = 1,$$

and, for $1 \leq k \leq n-1$,

$$P_{n,k} = P_{n-1,k-1} + P_{n-1,k}.$$

Although the rule is simple, several striking patterns already appear at this stage. The purpose of this article is to establish the first of them.

1. The boundary entries

The first visible fact is that every row begins and ends with 1.

Proposition 1. *For every $n \geq 0$, the first and last entries in row n are equal to* 1. *In symbols,*

$$P_{n,0} = P_{n,n} = 1.$$

Proof. This is built into the definition of the triangle. The top row consists of the single entry 1, and whenever a new row is formed, its boundary entries are taken to be 1. Hence the first and last entries in every row are 1. □

This fact may look too simple to deserve mention, yet it plays an important role. The boundary of the triangle acts as a fixed frame from which all interior entries are generated.

2. Symmetry of the rows

A second pattern is immediately apparent: each row reads the same from left to right as from right to left. For instance,

$$1,\ 4,\ 6,\ 4,\ 1$$

is symmetric, as is

$$1,\ 6,\ 15,\ 20,\ 15,\ 6,\ 1.$$

This is not a coincidence.

Proposition 2. *Each row of Pascal's triangle is symmetric. That is, for every $n \geq 0$ and every $0 \leq k \leq n$,*

$$P_{n,k} = P_{n,n-k}.$$

Proof. We proceed by induction on n.

For $n = 0$, the claim is trivial, since row 0 consists only of the single entry 1.

Assume now that row $n-1$ is symmetric. We show that row n is also symmetric.

The boundary entries satisfy

$$P_{n,0} = 1 = P_{n,n},$$

so symmetry holds at the ends.

Now let $1 \leq k \leq n-1$. By the construction rule,

$$P_{n,k} = P_{n-1,k-1} + P_{n-1,k}.$$

Using the inductive hypothesis, row $n-1$ is symmetric, so

$$P_{n-1,k-1} = P_{n-1,(n-1)-(k-1)} = P_{n-1,n-k},$$

and

$$P_{n-1,k} = P_{n-1,(n-1)-k} = P_{n-1,n-1-k}.$$

Therefore

$$P_{n,k} = P_{n-1,n-k} + P_{n-1,n-1-k}.$$

But the right hand side is precisely the rule for constructing the entry in position $n-k$ of row n, namely

$$P_{n,n-k} = P_{n-1,n-k-1} + P_{n-1,n-k}.$$

Since addition is commutative,

$$P_{n,k} = P_{n,n-k}.$$

Thus row n is symmetric. By induction, every row of Pascal's triangle is symmetric. □

Symmetry tells us that the left half of the triangle already determines the right half. In that sense, half the structure is a reflection of the other half.

3. The second diagonal and the counting numbers

Let us now move from rows to diagonals. Consider the shallow diagonal beginning with the second row from the left:

$$1,\ 2,\ 3,\ 4,\ 5,\ 6, \ldots$$

These are simply the counting numbers.

Proposition 3. *For every $n \geq 1$,*

$$P_{n,1} = n.$$

Thus the second shallow diagonal of Pascal's triangle consists of the counting numbers

$$1,\ 2,\ 3,\ 4,\ 5, \ldots$$

Proof. We again use induction on n.

For $n = 1$, row 1 is

$$1,\ 1,$$

so

$$P_{1,1} = 1.$$

Thus the statement holds.

Assume now that $P_{n-1,1} = n - 1$ for some $n \geq 2$. Then by the construction rule,

$$P_{n,1} = P_{n-1,0} + P_{n-1,1}.$$

Since $P_{n-1,0} = 1$, we obtain

$$P_{n,1} = 1 + (n-1) = n.$$

Hence the result holds for row n, and therefore for all $n \geq 1$. □

This proof shows something important. The counting numbers arise not by external insertion, but naturally from repeated addition of 1 along the edge.

4. The third diagonal and the triangular numbers

The next shallow diagonal is

$$1,\ 3,\ 6,\ 10,\ 15, \ldots$$

These are the triangular numbers. They are called triangular because the nth such number counts the dots required to form a triangular arrangement.

Proposition 4. *For every $n \geq 2$,*

$$P_{n,2} = 1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2}.$$

Thus the third shallow diagonal of Pascal's triangle consists of the triangular numbers.

Proof. We first note that, by the previous proposition,

$$P_{n-1,1} = n - 1.$$

We now prove the required formula by induction on n.

For $n = 2$, row 2 is

$$1,\ 2,\ 1,$$

so

$$P_{2,2} = 1.$$

Also,

$$1 + 2 + \cdots + (2-1) = 1.$$

Thus the formula holds at the starting point.

Assume now that for some $n \geq 3$,

$$P_{n-1,2} = 1 + 2 + \cdots + (n-2).$$

Then, using the construction rule,

$$P_{n,2} = P_{n-1,1} + P_{n-1,2}.$$

Substituting the known expressions,

$$P_{n,2} = (n-1) + \Big(1 + 2 + \cdots + (n-2)\Big).$$

Therefore,

$$P_{n,2} = 1 + 2 + \cdots + (n-1).$$

Using the standard formula for this sum,

$$P_{n,2} = \frac{n(n-1)}{2}.$$

Hence the third diagonal indeed consists of the triangular numbers. □

This result is particularly pleasing. The counting numbers form one diagonal, and the sums of the counting numbers form the next.

5. Higher diagonals and the simplex numbers

The same pattern continues. The diagonal after the triangular numbers is

$$1,\ 4,\ 10,\ 20,\ 35, \ldots$$

These are the tetrahedral numbers. The next one is

$$1,\ 5,\ 15,\ 35,\ 70, \ldots$$

These are called pentatope numbers. More generally, the shallow diagonals of Pascal's triangle produce a family of figurate numbers known as the simplex numbers.

To describe them clearly, it is convenient to define them recursively.

Let

$$S_0(n) = 1 \qquad \text{for all } n \geq 1,$$

and for each $d \geq 1$ define

$$S_d(n) = \sum_{r=1}^{n} S_{d-1}(r).$$

Thus:

$$\begin{aligned} S_1(n) &= 1 + 1 + \cdots + 1 = n, \\ S_2(n) &= 1 + 2 + \cdots + n, \\ S_3(n) &= \sum_{r=1}^{n} \frac{r(r+1)}{2}, \end{aligned}$$

and so on.

The numbers $S_1(n)$ are the counting numbers, $S_2(n)$ are the triangular numbers, $S_3(n)$ are the tetrahedral numbers, and $S_4(n)$ are the pentatope numbers.

The next proposition explains why these numbers appear in Pascal's triangle.

Proposition 5. *For each integer $d \geq 0$, the $(d+1)$st shallow diagonal of Pascal's triangle is given by*

$$S_d(1),\ S_d(2),\ S_d(3), \ldots$$

In other words, each shallow diagonal is obtained by summing the previous one term by term.

Proof. We argue by induction on d.

For $d = 0$, the first shallow diagonal consists entirely of 1's, which is exactly the sequence

$$S_0(1),\ S_0(2),\ S_0(3), \ldots$$

since $S_0(n) = 1$ for all n.

Now assume that the $(d+1)$st shallow diagonal is

$$S_d(1),\ S_d(2),\ S_d(3), \ldots$$

We must show that the next shallow diagonal is

$$S_{d+1}(1),\ S_{d+1}(2),\ S_{d+1}(3), \ldots$$

By the construction rule of Pascal's triangle, each entry of the next diagonal is obtained by adding two entries above it. One of these lies on the boundary or on the already formed part of the new diagonal, and the other lies on the previous diagonal. Consequently, as we move down the triangle, each new term in the next diagonal is obtained by adding the next term of the previous diagonal to the preceding term of the current diagonal. This means precisely that the new diagonal is formed by cumulative sums of the previous one.

Hence, if the previous diagonal is

$$S_d(1),\ S_d(2),\ S_d(3), \ldots,$$

then the next diagonal is

$$S_d(1),\ S_d(1) + S_d(2),\ S_d(1) + S_d(2) + S_d(3), \ldots$$

That is,

$$S_{d+1}(1),\ S_{d+1}(2),\ S_{d+1}(3), \ldots$$

by the recursive definition of S_{d+1}.

Therefore the claim holds for all $d \geq 0$. □

The importance of this proposition is not merely terminological. It shows that Pascal's triangle contains, in a systematic way, the repeated sums of repeated sums of 1. This is why so many familiar number sequences appear inside it.

Remark 1. *The first few shallow diagonals may therefore be listed as follows:*

$$1,\ 1,\ 1,\ 1, \ldots$$

$$1,\ 2,\ 3,\ 4, \ldots$$

$$1,\ 3,\ 6,\ 10, \ldots$$

$$1,\ 4,\ 10,\ 20, \ldots$$

$$1,\ 5,\ 15,\ 35, \ldots$$

That is, boundary entries, counting numbers, triangular numbers, tetrahedral numbers, and pentatope numbers.

Conclusion

Even before one studies its algebraic significance, Pascal's triangle already reveals a remarkable internal order. Its boundary remains fixed at 1, each row is symmetric, and its shallow diagonals generate whole families of figurate numbers. None of these patterns is accidental. Each arises directly from the basic rule by which the triangle is formed.

In the next article, we shall move from visible structure to algebraic meaning, and show that the rows of Pascal's triangle are precisely the coefficients that appear in the expansion of $(1 + x)^n$.